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Geometrical Basis for the Standard Model

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The robust character of the Standard Model is confirmed. Examination of its geometrical basis in three equivalent internal symmetry spaces—the unitary plane C^2, the quaternion space Q, and the real space R^4— as well as the real space R^3 uncovers mathematical properties that predict the physical properties of leptons and quarks. The finite rotational subgroups of the gauge group $SU(2)_L \times U(1)_Y$ generate exactly three lepton families and four quark families and reveal how quarks and leptons are related. Among the physical properties explained are the mass ratios of the six leptons and eight quarks, the origin of the left-handed preference by the weak interaction, the geometrical source of color symmetry, and the zero neutrino masses. The (u, d) and (c, s) quark families team together to satisfy the triangle anomaly cancellation with the electron family, while the other families pair one-to-one for cancellation. The spontaneously broken symmetry is discrete and needs no Higgs mechanism. Predictions include all massless neutrinos, the top quark at 160 GeV/c^2, the b' quark at 80 GeV/c^2, and the t' quark at 2600 GeV/c^2.

1. INTRODUCTION

For at least 20 years the successful predictions of the minimal Standard Model (Glashow, 1961; Weinberg, 1967; Salam, 1968; Kim, 1990; Langacker, 1989) of leptons and quarks have been truly remarkable. Based upon the direct product gauge group $SU(3)_C \times SU(2)_L \times U(1)_Y$, the leptons and quarks occupy weak isospin doublet states, the quarks display three values of color charge, and the 12 gauge bosons have the correct physical properties. The theory has been resilient to attack from all sides, rejecting attempts (Renton, 1990) to embed it in a larger gauge group such as $SU(5)$ or to extricate a component structure (Rosner and Worah, 1992). In all cases, the empirical evidence places severe restrictions against the likelihood that any of these proposed schemes will prove successful. Yet, in spite of all the successes of the minimal Standard Model, there exists an

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uneasiness about a theory that cannot predict the masses of its fundamental particles nor dictate the reason for having at least three families of leptons and quarks. Too many unanswered questions remain for the theory to be considered complete as presently understood.

Under these circumstances, I am proposing a different approach (Potter, 1989) which is aimed at achieving a better understanding of the geometrical properties of the Standard Model when its gauge group operates in the unitary plane. The approach reveals some remarkable mathematical properties which have been ignored for over 20 years, properties which resolve many of the pertinent physics questions about leptons and quarks, including the mass spectrum, the family problem, the origin of color symmetry, the fundamental difference between leptons and quarks, and the reason for a left-handed preference by the weak interaction. One discovers that the Standard Model is such a good first approximation to the ultimate truth about leptons and quarks that the only major modification required to improve significantly our understanding is to assign the lepton and quark weak isospin families to *finite rotational subgroups* of the electroweak gauge group $SU(2)_L \times U(1)_Y$ of the theory. By using this geometrical approach, one realizes that the unitary plane C^2 of operation of the electroweak gauge group must also be treated in terms of its equivalent spaces, the quaternion space Q and the 4-dimensional real space R^4. Furthermore, one finds that the 3-dimensional real subspace R^3 is important for understanding the whole fermion family hierarchy within the framework of the Standard Model.

A careful accounting of the gauge group operating in these spaces leads to the grand prediction of exactly three lepton families and four quark families, each family corresponding to a finite rotational subgroup of $SU(2)_L \times U(1)_Y$ in R^3 and R^4, respectively. This grand prediction does not disagree with the empirical results (Abrams *et al.*, 1989; ALEPH Collaboration, 1989; OPAN Collaboration, 1989; DELPHI Collaboration, 1989) that dictate three lepton families with light neutrinos if one recalls that these results do not actually limit the number of quark families. In addition, the triangle anomalies can be shown to cancel still, even though there is a mismatch of family numbers. Some predicted numerical quantities are the quark masses, including the top quark at about 160 GeV/c^2, the fourth family b' quark at about 80 GeV/c^2, and the t' quark at about 2600 GeV/c^2. The existence of the b' quark becomes the acid test for this modification of the minimal Standard Model because it represents a fourth quark family, and its predicted mass value at about 80 GeV/c^2 means that it should have been produced at Fermilab (Agrawal and Hou, 1992; Agrawal, Ellis, and Hou, 1990).

I call attention first to the mismatch of three lepton families to four quark families because one suspects that the triangle anomalies (Adler, 1969; Bell and Jackiw, 1967) will not cancel as they do when the number of quark families matches the number of lepton families. Indeed, this objection is an important concern, but it can be resolved, as I show in Section 2. The transformation properties of the particle states in the unitary plane are discussed in Sections 3–7 where the left-handed preference of the weak interaction is explained, the origin of $SU(3)$ color is proposed, the differences between quarks and leptons are examined, and the argument for zero neutrino masses is given. Properties of the finite rotation groups are introduced and exploited in Sections 8–10, in which their invariant polynomial bases are discussed, mass ratios for the leptons are shown to arise from 3D rotational group invariants, the 4D rotational groups are related to the 3D rotational groups and to quarks, and the mass hierarchy for quarks is resolved. In Section 11 a new expression for electric charge is introduced, and in Section 12 the symmetry breaking of the gauge group is proposed to be discrete with no need for Higgs bosons. Some final comments in Section 13 recap the major ideas and consequences.

2. THE TRIANGLE ANOMALY CANCELLATION

The modification of the minimal Standard Model to be introduced in later sections predicts four quark families and three lepton families, a mismatch of family numbers that seems to create a problem with the cancellation of the triangle anomalies. To allay these concerns, I choose to discuss this conflict first to illustrate that the standard cancellation procedure remains successful when the first two quark families are paired with the electron family. The application to the complete quark family hierarchy will be discussed at the end of this section.

The triangle anomaly arises from graphs of the kind shown in Fig. 1. Diagrams (a) and (c) involve the first family of quarks only; diagrams (b) and (d) are the corresponding diagrams in which the charm and strange quarks of the second family substitute between vertices b and c. There are ten generic graphs in total: the four in Fig. 1 for the first quark family, the four analogous graphs for the second quark family, and the two graphs for the electron family analogous to (a) and (c). There are many processes which have triangle graphs involving the weak interaction, but the same type of cancellations occur in every example (Ryder, 1985), so one can use the representative diagrams shown. These processes involve high-order diagrams which have very small contributions to the overall amplitude, but they must cancel exactly in order to preserve the renormalizable theory of the weak interaction.

By following the standard calculation procedure, the important expression for the left-handed fermions interacting via the Z° and the W's at the

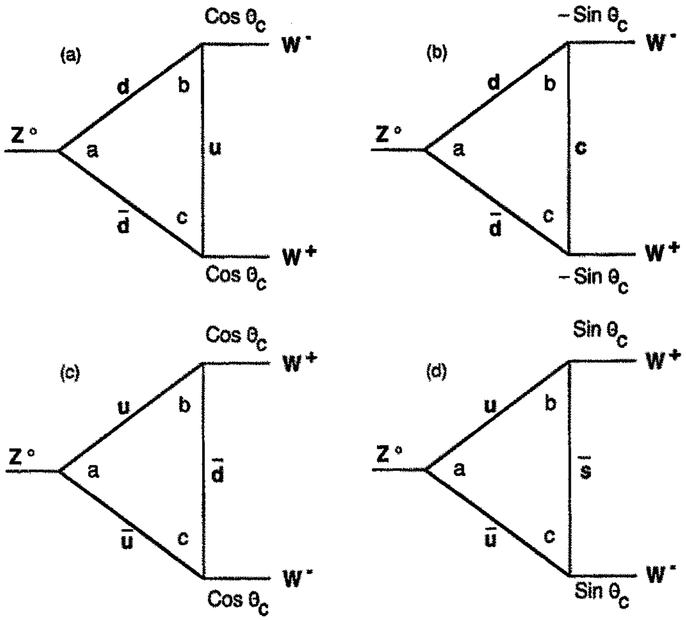

Fig. 1. Triangle diagrams for $Z^\circ \to W^+W^-$ via (a) u-quark exchange, (b) c-quark exchange, (c) d-quark exchange, and (d) s-quark exchange, for the (u, d) quark family.

three triangle diagram vertices becomes

$$\mathrm{Tr}(I_3 + \sin^2\theta_w\, Q)\{M_b, M_c\} = 0 \tag{1}$$

where the first factor $(I_3 + \sin^2\theta_w\, Q)$ is the contribution from the Z° coupling at vertex a and the second factor involves the vertex factors M_b and M_c. The fermion weak isospin is I_3, θ_w is the Weinberg angle, and Q is the electric charge of the left-handed fermion. The vertex factors $M = gI_+$ or $M = gI_-$ depend upon the W boson at vertex b or c via the weak isospin raising and lowering operators I_+ and I_- and the weak coupling constant g. One should multiply the standard coupling constant g by the Cabibbo factor for the interactions of the (u, d) and (c, s) families at the vertices b and c involving the W boson, so I will rewrite g as g' to indicate the inclusion of the Cabibbo factors. One takes $g' = g$ for leptons, $g' = g \cos\theta_c$ at a "u, d" vertex, and $g' = -g \sin\theta_c$ or $+g \sin\theta_c$ at a "c, d" or "u, s" vertex. Since $\{I_+, I_-\} = 1$ and the $\mathrm{Tr}\, I_3 = 0$ for each family, the condition in equation (1) reduces to

$$\sum_i Q_i g'_b g'_c = 0 \tag{2}$$

with the subscripts b and c again identifying the triangle vertices.

By first ignoring the Cabibbo factors at the vertices, one can reproduce the standard result $\sum_i Q_i = 0$ obtained when the family pairing is normal, i.e., electron family paired to the up/down family as $(\nu_e, e) \leftrightarrow (u, d)$:

$$\sum_i Q_i = Q_\nu + Q_e + 3(Q_u + Q_d) = 0 + (-1) + 3(2/3 - 1/3) = 0 \qquad (3)$$

When the Cabibbo factors are included in the vertex terms for the first quark family (u, d), then the cancellation will not occur because the factor $\cos^2\theta_c$ arising from diagrams (a) and (c) in Fig. 1 multiplies the quark contribution term in equation (3). Thus the normal family pairing scheme in the Standard Model does not produce the required cancellation!

The modified minimal Standard Model introduces a different family pairing scheme than the one traditionally proposed. The pairings dictated by the finite rotational group properties are the seemingly bizarre arrangement with the first quark family left unpaired to its own lepton family:

leptons		quarks
		(u, d)
(ν_e, e)	$\leftrightarrow$	(c, s)
(ν_μ, μ)	$\leftrightarrow$	(t, b)
(ν_τ, τ)	$\leftrightarrow$	(t', b')

The unpaired (u, d) quark family looks odd and ruins the anticipated one-to-one matching, but this bizarre scheme of relationships between lepton and quark families may very well be the correct one because the scheme leads to exact cancellation. If one combines the first two quark families with the electron family in order to evaluate equation (2) for all ten graphs, the (u, d) charge contributions from (a) and (c) types multiplied by $\cos^2\theta_c$ add to the (c, s) charge contributions from (b) and (d) types multiplied by $\sin^2\theta_c$, so that every piece contributes to make the sum add to zero exactly:

$$\begin{aligned} &Q_\nu + Q_e + 3(Q_c + Q_s) + 3(Q_u + Q_d) \\ &\quad = 0 + (-1) + 3(2/3 - 1/3)[\sin^2\theta_c + \cos^2\theta_c] = 0 \qquad (4) \end{aligned}$$

Remarkably, it is the mixing of the (u, d) family and the (c, s) family in this bizarre pairing scheme with the electron family that is required to achieve the cancellation. If there were no mixing of the (u, d) and (c, s) quark families, there would be no cancellation of the triangle anomalies for the electron family.

Of course, in the modified version, the two remaining sets of paired families of leptons and quarks will cancel in the usual one-to-one manner, i.e., (ν_μ, μ) with (t, b) and (ν_τ, τ) with (t', b'), if there is no further mixing between quark families. If further mixing does exist so that the Cabibbo–Kobayashi–Maskawa matrix is peppered with nonzero off-diagonal terms beyond the Cabibbo terms, then all the detailed couplings at the vertices of the triangle diagrams would need to be calculated to ensure cancellation. Fortunately, examination of the geometrical basis (in a later section) uncovers prospects for mixing between the first two quark groups only. Therefore, in the modified minimal Standard Model with four quark families and three lepton families the triangle anomalies cancel exactly.

3. PARTICLE STATES IN THE UNITARY PLANE

I am going to assume that the mathematical properties of the unitary plane dictate the physical properties of the fundamental fermions. Evaluations of this assumption will be made at the appropriate places. One begins with the minimal Standard Model which assigns the two fundamental fermions in each lepton family and in each quark family to weak isospin $SU(2)_L$ doublets. These particle states correspond to the two complex basis spinors u and v which span the two-dimensional complex space $\langle u, v|$, i.e., the unitary plane C^2, in this two-dimensional representation of the gauge group. The corresponding two fundamental antifermions are assigned to the two basis spinors $-u^*$ and v^* which span the complex conjugate space $\langle v^*, -u^*|$. An important property of the electroweak direct product group $SU(2)_L \times U(1)_Y$, where the Y refers to weak hypercharge, is that the two spinor spaces $\langle u, v|$ and $\langle v^*, -u^*|$ are inequivalent. [This inequivalence condition is in sharp contrast to their equivalence for the Lie group $SU(2)$ alone.] These two inequivalent spaces $\langle u, v|$ and $\langle v^*, -u^*|$ allow the four *distinct* particle states to be defined, the two fermion states in $\langle u, v|$ and the two corresponding antifermion states in $\langle v^*, -u^*|$. In the electron family, for example, the lepton states are the electron neutrino and the electron, while the antilepton states are the positron and the electron antineutrino. Here I emphasize that the neutrino and antineutrino are taken to be distinct Dirac particles with zero mass, and I give the arguments in a later section to support this assertion.

4. WEAK ISOSPIN LEFT–RIGHT DICHOTOMY

The weak interaction prefers the left-handed weak isospin states. Indeed, numerous empirical observations (Langacker, 1989, 1992) verify that only the left-handed fermion doublets participate in the weak interactions mediated by the W and Z° bosons. The right-handed fermions act as $SU(2)_L$ singlets and do not interact with the W or the Z°, but they participate in the electromagnetic interaction. Moreover, the experiments have revealed that the direct product group $SU(2)_L \times U(1)_Y$ is the weak isospin-hypercharge group for leptons and quarks or, at the very least, is an excellent approximation to the true group.

What type of spinor dichotomy could be the source of the left-handed preference? Or, expressed another way, why left-handed doublets and right-handed singlets? There are known to be two types of spinor dichotomy: (1) the dotted and undotted spinors that behave differently under the action of the Lorentz transformations of the Poincaré group, and (2) the spinors in the unitary plane of the internal symmetry group behaving differently under general unitary transformations in the plane itself. Although the former dichotomy is well known in physics, the latter was elucidated (Crowe, 1961; Coxeter, 1974) for the first time in the early 1960s and seems to have remained unused in particle physics.

Recall that the first type of left–right spinor dichotomy arises by considering the behavior of spinors under the action of the Lorentz transformation. The familiar dotted and undotted spinors appear and they correspond to the left-handed and right-handed spinor helicity states, respectively. This specific left–right dichotomy is independent of the interaction type because it arises from kinematic considerations alone and cannot be the ultimate origin of the left-handed preference of the weak interaction. The Lorentz transformation simply separates out the left-handed and the right-handed helicity states from the total collection which must already contain both kinds.

The origin of the weak isospin preference actually lies in the second type of dichotomy behavior. The particle states, i.e., the basis spinors u and v, when acted upon by the weak bosons to transform the initial weak isospin state into the final weak isospin state, undergo left and right screw transformations in the unitary plane. In order to appreciate these transformations, one can follow standard mathematical methods (Crowe, 1961; DuVal, 1964; Coxeter, 1974). The arbitrary point (u, v) in the unitary plane C^2 is transformed by the general $SU(2)$ matrix into (u', v') via

$$(u', v') = (u, v)\begin{pmatrix} \varepsilon a & \varepsilon c \\ -\varepsilon c^* & \varepsilon a^* \end{pmatrix} \tag{5}$$

which identifies $u' = \varepsilon(au - c^*v)$ and $v' = \varepsilon(cu + a^*v)$. This transformation preserves $uu^* + vv^*$, while the complex numbers e, a, and c obey $\varepsilon\varepsilon^* = 1$, $aa^* + cc = 1$, and $\varepsilon^2 = \exp(i\phi)$.

At this point, one could simply factor out the ε from the matrix and let it operate as a phase factor on the left side of (u, v) in equation (5), but better insight is gained by expressing this general unitary transformation in terms of quaternions. The point (u, v) is the quaternion

$$X = u + v\mathbf{j} \tag{6}$$

with $\mathbf{j}$ the unit imaginary just like $\mathbf{i}$. Quaternions do not commute, so $\mathbf{ij} = -\mathbf{ji}$, etc. The product of two quaternions produces

$$(u + v\mathbf{j})(a + c\mathbf{j}) = au - c^*v + (cu + a^*v)\mathbf{j} \tag{7}$$

which looks similar to the result of the transformation to the u' and v' in equation (5). The general unitary transformation is then expressed via quaternions as the product

$$X' = \varepsilon X \kappa \tag{8}$$

where $\kappa = a + c\mathbf{j}$, a quaternion of norm one, and ε is a unit quaternion also.

By comparison to the matrix format, κ is the $SU(2)$ matrix and the general transformation becomes the expected result

$$(u', v') = \varepsilon(u, v)\begin{pmatrix} a & c \\ -c^* & a^* \end{pmatrix} \tag{9}$$

which must be interpreted in terms of quaternions in order to realize the connection to doublets and singlets. First, one needs to know that the left-multiplication by the unit quaternion ε is mathematically a *right screw* transformation that will be selected out into a right-handed helicity state when the Poincaré transformation is applied. The right-multiplication by the unit quaternion κ is the *left screw* transformation that will be selected out as a left-handed helicity state. For reference purposes, in the complex conjugate unitary space $\langle v^*, -u^*|$ the quaternion κ^* is necessary and produces a right screw transformation, etc.

Where do the doublet and singlets originates? The left-handed doublet behavior arises because the left screw κ acts on (u, v) to produce $(au - c^*v, cu + a^*v)$, an entity that behaves as a doublet. The singlet arises because the right screw (unit quaternion) ε acts on (u, v) to produce $(\varepsilon u, \varepsilon v)$, i.e., u and v transform as singlets simply multiplied by the unit quantity ε. Thus, transformations in the unitary plane involve only left-handed doublets and right-handed singlets. There are no exceptions! One can verify that the handedness correlation reverses in the conjugate spinor space.

The two types of left–right dichotomy listed earlier are related even though the first type is the result of a space-time transformation and the second type is the result of a transformation in the unitary plane of the internal symmetry space. The internal-space transformation tells us that the spinors are divided into two types by the quaternion screw transformations, and the external space-time transformation simply separates these two types into left- and right-handed helicity states.

The physical consequences of the general unitary transformation are enormous. For example, if one is interested in left–right symmetric models based upon $SU(2)_L \times SU(2)_R$ weak isospin symmetry, the analysis for the general transformation proceeds in exactly the same manner as the one above because the original unitary space for $SU(2)$ is still the space for the left–right direct product group. The immediate consequence is left-handed doublets and right-handed singlets because one cannot force the mathematical properties to produce a true left–right symmetry for the general transformation in the unitary plane. Consequently, the left–right symmetric models exist in name only, do not produce true left–right symmetry mathematically, and should not exist physically in nature. Proponents of these left–right models propose a W_R, a "right-handed" W, which they can argue to be very massive in order to have escaped experimental detection. But the argument discussed above eliminates a W_R at any mass value.

The W_R is very important because its existence would verify the presence of a right-handed weak current (RHC) and indicate that a left–right symmetric model must be taken seriously. However, the W_R has an even more important role, for its existence would signify that the physical properties of the weak eigenstates of the Standard Model are *not* dictated by the mathematical properties of $SU(2) \times U(1)_Y$ in the unitary plane. In contrast, the absence of the W_R and its RHC would eliminate left–right symmetric models and would support the assumption that the mathematical properties of the unitary plane C^2 dictate the physical properties of the fundamental fermions.

What is the evidence with regard to a W_R? No RHC has been found, and lower limits on the mass of the W_R have been given. The recent searches (Carnoy *et al.*, 1990; Dubbers *et al.*, 1990; Jodido *et al.*, 1986; Stoker *et al.*, 1985) for right-handed weak currents involving the second weak boson W_2 as predicted by the left–right symmetric model when applied to the K^+ and μ^+ decays has put a constraint on the lower mass limit of the $W_2 \gg 653$ GeV at the Cabibbo angle value for the right-handed sector defined by $|\sin\theta_R| = 1$. W_2 is predominantly right handed ($W_2 = W_L \sin\zeta + W_R \cos\zeta$, $|\zeta| \ll 1$, and W_L and W_R are the boson weak eigenstates and ζ is the mixing angle) and has been proposed to possess a much heavier mass than does the predominantly left-handed boson W_1 ($W_1 = W_L \cos\zeta - W_R \sin\zeta$). Other experimental searches involving a number of processes, including muon-decay positron asymmetry, have been done with high precision and they establish a lower limit mass value for the $W_2 > 482$ GeV at $\zeta = 0$.

In summary, for the case of weak isospin in the unitary plane, it appears that nature had no choice but to divide the weak isospin states into left-handed doublets and right-handed singlets. The physical consequences of this dichotomy first showed up in the 1950s as the violation of parity for the weak interaction because only left-handed fermions participated. With the mathematical elimination of a W_R (and with no empirical evidence for a W_R), I will continue to assume that the physical properties of the fundamental fermions are dictated by the mathematical ones. Knowing that the doublet–singlet dichotomy simply expresses the general unitary transformation in C^2, one can make a few general predictions:

(a) All fermion weak isospin states will act as left-handed doublets and right-handed singlets.
(b) One can eliminate a very massive right-handed W boson which would interact with the weak isospin states in the unitary plane.
(c) There will be no heavy right-handed partners, too heavy to be observed in the relevant experiments, for the right-handed fermions to form a right-handed $SU(2)$ doublet.
(d) There must exist the ν_τ and the top quark in order to form the left-handed weak isospin doublets.

In this section I have shown that the mathematical properties of the general transformation in the unitary plane dictate no other possibility than left-handed doublets and right-handed singlets. As a consequence, there is no need to label explicitly the left-handedness of the electroweak group as $SU(2)_L \times U(1)_Y$. In addition, the resolution of this one problem suggests that other enigmas related to the Standard Model may be amenable by a better understanding of the mathematical properties of its gauge group operating in the unitary plane. The ultimate goal would be to explain all the physical properties of the leptons and quarks as arising from the mathematical ones.

5. POSSIBLE ORIGIN OF *SU*(3) COLOR

One more aspect of the minimal Standard Model needs consideration in order to complete the framework for a discussion of the finite rotational subgroups of the direct product gauge group $SU(3)_C \times SU(2)_L \times U(1)_Y$. Color symmetry associated with the Lie group $SU(3)_C$ was proposed to preserve the antisymmetric wavefunctions for fermions, but its ultimate origin has never been identified. I propose that color symmetry simply represents the partitioning of the 4D real space into its three sets of equivalent 4D rotation-plane pairs. This identification, if correct, then dictates an inherent difference between leptons and quarks because the quark states can be shown to occupy the whole 4D space (in order to have color charge) while the lepton states span the 3D subspace only (in order to not have color charge).

The key mathematical ideas utilize the relationships among four vector spaces. The unitary plane C^2 has intimate connections to three other spaces: the Euclidean spaces R^4 and R^3 and the quaternion space Q. The spaces R^4 and Q are equivalent spaces to C^2, while R^3 is a subspace. Rotations defined in R^4 and Q have corresponding rotations defined in C^2, and rotations in C^2 are 2-to-1 homomorphic to rotations in R^3. One needs all these relationships to understand how the particle states $\langle u|$ and $\langle v|$ behave.

The connection between the point (u, v) in the unitary plane C^2 to the quaternion $q = u + v\mathbf{j}$ in Q has already helped explain the origin of the left-handed preference for the weak interaction. One can now expand the list of spaces (DuVal, 1964; Coxeter, 1974) to include the four-dimensional space R^4 via

$$q = u + v\mathbf{j} = w + x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \tag{10}$$

with $u = w + x\mathbf{i}$ and $v = y + z\mathbf{i}$, where $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are the unit imaginaries and w, x, y, z are real numbers. For completeness, one also needs the product of the quaternion q with another quaternion $q' = (w', x', y', z')$,

$$\begin{aligned} qq = (&ww' - xx' - yy' - zz', wx' + xw' + yz' - zy', \\ &wy' - xz' + yw' + zx', wz' + xy' - yx' + zw') \end{aligned} \tag{11}$$

in order to work in either space, Q or R^4. From the space R^4 one can identify the familiar 3-dimensional subspace R^3, called the imaginary prime, defined by fixing the value of w and allowing variation in x, y, and z. More generally, one could define a 3-vector about which rotations in R^3 can occur.

The gauge group operations in the 4-dimensional real space R^4 are represented by a 4×4 matrix. For example, the $SU(2)$ matrix defined in the unitary plane C^2 can be written both as the unit quaternion $q = (w, x, y, z)$ in Q with norm $w^2 + x^2 + y^2 + z^2 = 1$ and as the 4-dimensional rotation matrix of determinant unity which has the block diagonal form for the ordered axes set (w, x, y, z)

$$\begin{bmatrix} \cos\alpha & \sin\alpha & 0 & 0 \\ -\sin\alpha & \cos\alpha & 0 & 0 \\ 0 & 0 & \cos\beta & \sin\beta \\ 0 & 0 & -\sin\beta & \cos\beta \end{bmatrix} \tag{12}$$

One could express the α and β in terms of the a, a^*, c, and c^* of the $SU(2)$ matrix in (5) when needed. This specific 4×4 matrix form emphasizes that the 4D rotation always occurs in two orthogonal planes, here in the (wx) plane and in the (yz) plane. These planar rotations are two separate rotations that commute, and when $\alpha - \beta = 0$, the 4D rotation is the right screw transformation. When $\alpha + \beta = 0$, the 4D rotation is the left screw transformation. These are the screw transformations referred to earlier in the discussion of the doublet–singlet dichotomy.

The 3D rotation in the imaginary prime (**i**, **j**, **k**) can be expressed in terms of the left and right screw transformations also. If the quaternion $p = (\cos\alpha, \sin\alpha, 0, 0)$, then the right screw transformation $X \to pX$ is the compound rotation of angle α in the (w, x) plane and angle α in the (y, z) plane, and the left screw transformation $X \to Xp^{-1}$ is the compound rotation of angle $-\alpha$ in the (w, x) plane and the angle α in the (y, z) plane. The resultant $X \to pXp^{-1}$ is the rotation by 2α in the (y, z) plane, i.e., a rotation in the imaginary prime 3D subspace about the x axis.

The 4D rotation matrix given above in (12) involves a pair of orthogonal planes in the real space R^4. Further examination of 4D rotations reveals the existence of *three distinct pairs* of orthogonal planes in R^4: $[wx, yz]$, $[xy, zw]$, $[yw, xz]$. The rotations can be expressed using any one of these three pairs of planes because the pairs are equivalent, which is just another way to state that there is a symmetry among the three pairs.

I propose that this symmetry among the three pairs of planes is the origin of $SU(3)_C$ color symmetry for the quark states. The $SU(3)$-like properties can be mathematically checked, for one can show by multiplying the matrices that the Lie group associated with their symmetry is $SU(3)$. If one assigns the colors red, green, and blue, respectively, to these three pairs of planes, two particular combinations of matrix products lead to no net rotation, which are, when stated in terms of color combinations: color with anticolor (a quark with an antiquark), and the hadron composed from each of the three colors (three quarks) or each of the three anticolors (three antiquarks). In order to verify the $SU(3)$ behavior, one must include these special combinations of rotations which produce no net 4D rotation.

In addition to verifying the $SU(3)$ behavior, one learns also that the two-color combination red-green, for example, is not the anti-blue color state, this latter designation being used often in the literature. The anti-blue designation should really be reserved for the complex conjugate spinor space and the 4×4 matrix representing this 4D rotation.

Mathematically, when this "new" color symmetry is included in the operations in the unitary plane, the $SU(2)_L \times U(1)_Y$ group for the electroweak interactions is extended by the $SU(3)_C$ to become the direct product group $S(3)_C \times SU(2)_L \times U(1)_Y$ of the Standard Model. This extension obtained by taking the direct product of groups is analogous to the direct product group of the square in a real plane. First consider the rotation group C_4 of a square in the real plane and then add the mirror reflection group σ_2 to make the direct product group $\sigma_2 \times C_4$. This direct product group of the square is not isomorphic to any other single group G.

Analogously, the direct product group $S(3)_C \times SU(2)_L \times U(1)_Y$ for leptons and quarks also may not be isomorphic to any other group.

If the identification of the color symmetry with the three distinct sets of 4D rotation planes is correct, then some insight into the similarities and differences between leptons and quarks can be coaxed from the mathematical properties. In the next section the mathematical properties of the various spaces are exploited once more to find the most likely space for the leptons so that they possess no color charge yet fit into the finite subgroup scheme to be developed in a later section.

6. MATHEMATICAL DIFFERENCE BETWEEN LEPTONS AND QUARKS

What makes quarks different fundamentally from leptons? This question is equivalent to asking why quarks have baryon number $B \neq 0$ and leptons have $B = 0$. The question can be answered by examining the different dimensions of the real spaces required by the quarks and by the leptons to define their weak isospin particle states.

The color charge of the quark states requires the whole 4D real space in order to have rotations occur in two orthogonal planes. Because experiment has confirmed that the leptons are colorless, one can conclude that the lepton particle states in each weak isospin family do not span all of the R^4 vector space, otherwise leptons would have color charge also. At most, the lepton states can span the subspace R^3, the imaginary prime. I will assume that the lepton states do span R^3 and determine the consequences.

The color charge distinction between spaces makes quarks 4-dimensional geometrical entities and leptons 3-dimensional geometrical entities. Baryon number could conveniently represent the same information. In addition, the inability to isolate a single quark, i.e., color confinement, may have its origin in the 4D character of the quarks, but I have no further insight into this particular physical behavior of the quarks.

Instead, I direct attention to the three equivalent internal spaces C^2, R^4, and Q in which vectors of norm unity and transformation matrices A with $|\det A| = 1$ are the fundamental mathematical objects. These restrictions suggest the consideration of points on the appropriate unit hypersphere. A rotation in R^4 transforms an arbitrary point (w, x, y, z) to another point in R^4 or, equivalently, the point on the unit hypersphere S^3 to another point on S^3. The two basis spinors u and v corresponding to the particle states are points on S^3 each defined by three parameters. That is, each quark particle state is defined by three parameters in the internal symmetry space R^4.

In the lepton case, the spinors are confined to the subspace R^3. A rotation transforms the point (x, y, z) to another point in R^3 or, equivalently, the point on the sphere S^2 to another point on S^2. The two basis spinors u and v for leptons are restricted now to two parameters each which define their points on S^2.

In space-time, the connection between the mass of a particle and the number of degrees of freedom for its spin vector falls into two categories. A particle with mass must have three degrees of freedom so that its spin can point in any one of the three space directions in space-time. A zero-mass particle cannot have more than two degrees of freedom, each degree of freedom corresponding to one helicity state. In addition, a zero-mass particle could have only one degree of freedom and one helicity state under certain restrictions.

Is there a direct relationship between the number of parameters needed to define the basis vectors in the internal symmetry space and the number of possible spin directions in space-time? Yes, the two quantities are equal. A particle state defined by three parameters on S^3 will have three degrees of freedom for its spin vector. A particle state defined by two parameters on S^2 will have two degrees of freedom for its spin vector. In terms of fermion states, therefore, the two basis spinors on S^3 would correspond to two massive fermion states, i.e., the quark states. One would expect also that the two basis spinors confined to S^2 represent two massless fermions, the lepton states. Further discussion of the lepton states is reserved for the next section.

I have proposed that the two quark states in each quark family span the whole space R^4 and two lepton states in each lepton family span the subspace R^3 in order to agree with the color charge assignments. As a consequence, baryon number becomes a bookkeeping quantity for 4D entities called quarks and antiquarks. Because baryon number is thought to be a conserved quantum number, an associated symmetry operation is expected, but its identification remains obscure. One test of baryon number conservation is proton decay, which has been predicted by several models. If a proton does decay, then baryon number conservation is violated because at least one of the up or down quarks in the proton must have changed into nonquark entities which would have $B = 0$. So far, there is no evidence for proton decay. There should be no proton decay in the modified minimal Standard Model because the fundamental gauge group remains as a direct product group so that no leptoquark-like particles appear.

7. ZERO-MASS NEUTRINOS

The quark and lepton assignments made in the previous section are very interesting because nature would have two massive quark states per family and two massless lepton states per family. Certainly, the quark assignments to the basis vectors on S^3 lead to agreement with the empirical results because the quarks have color charge and nonzero mass. But the lepton assignments would be wrong because both leptons in a family do not possess zero mass. Apparently, nature has chosen an alternative way for the two lepton states to exist in R^3. The production of one massive lepton and one massless lepton in each family requires the breaking of the symmetry between the two massless states on S^2, with the result that the four total parameters defining the two particle states is reapportioned into a lepton with mass (three parameters) and a massless lepton with only one helicity state (one parameter).

Whether this symmetry-breaking scenario is the real origin of the two observed lepton states remains an open question, but the proper lepton states are produced, one with mass and the other massless. Furthermore, this scenerio could possibly fit within the inherent symmetry-breaking behavior in the minimal Standard Model with or without a Higgs particle.

If the suggested scenario is accepted, one has accessible now a possible resolution of the neutrino mass problem. With the electron state being massive and requiring three parameters, the one remaining parameter available for the neutrino state forces the neutrino in each lepton family to be massless and to possess only one helicity state. Indeed, the mathematical argument eliminates the possibility for neutrinos to acquire a nonzero mass by any mechanism.

There is another important consequence of this symmetry breaking scenario. The existence of one massive lepton per family, the electron, for example, is the guarantee of charge-changing weak interactions for both the electron and the neutrino states with the W bosons, a process that would have been prohibited with only two massless lepton states because electric charge conservation would be violated. And, presumably, this fermion symmetry breaking happened simultaneously with the standard gauge symmetry breaking that produced the massive weak bosons. Then all the massless leptons produced before the symmetry breaking occurred would have intrinsic value in a residual way, for they should be excellent dark matter candidates because they can interact via the neutral current and gravitational interactions only!

8. THE FINITE ROTATIONAL SUBGROUPS OF $SU(2) \times U(1)_Y$

The investigation into the geometrical properties of the unitary plane C^2 and its equivalent spaces R^4 and Q has already revealed that some of the known fundamental fermion properties are dictated very nicely by the geometry. For example, the mathematics strongly suggests that the lepton states u and v in the unitary plane correspond to basis states that span the 3D subspace R^3 and that the quark states correspond to the basis states that span the whole space R^4. But the geometrical picture remains incomplete because we have yet to identify the origin of the different fermion families as well as the origin of the particle masses.

There has been no easy path pointing directly to the origin of the particle families and their pairings into generations. Any model, including the minimal Standard Model, requires the pairing of quark families and lepton families so that the triangle anomalies cancel, as we have seen in an earlier section. Without additional knowledge beyond the minimal Standard Model, one realizes that the quark-lepton family pairings can be made in several different ways and still satisfy the demands of the empirical results. For example, the (u, d) quark family could be paired with the (μ_τ, τ) family. And, most frustrating of all, there seem to be no obvious simple relationships among the particle masses to indicate the direction to pursue.

I have determined that the key idea missing from the minimal Standard Model is the connection between the fundamental fermions and the finite rotational subgroups of $SU(2) \times U(1)_Y$. This deliberate change in viewpoint from the continuous Lie groups to the finite rotational subgroups brings about a remarkable change in the understanding of the fundamental fermions within the general framework of the minimal Standard Model. One begins by considering the group $SU(2) \times U(1)_Y$ (without the L) and its isomorphism to the unit quaternion group $\mathbf{Q}$ that consists of all unit quaternions in R^4. In fact, one could better define $\mathbf{Q}$ in terms of its isomorphism (Altmann, 1986) to the group $SU'(2) = SU(2) \times C_i$, a form which explicitly shows the two-element inversion group C_i as a component. That is, $\mathbf{Q}$ and $SU'(2)$ contain matrices of determinant $+1$ and -1. These group relationships hint that $U(1)_Y$ will turn out to be isomorphic to C_i in the final complete model.

The next step utilizes the finite subgroups of the quaternion group $\mathbf{Q}$ which are labeled $\langle p, q, r\rangle = (p, q, r) \times C_i$, where the (p, q, r) are the familiar proper rotation groups in R^3. Some properties of the finite subgroups (DuVal, 1964; Coxeter, 1974) of $\mathbf{Q}$—C_n, $\langle p, 2, 2\rangle$, $\langle 3, 3, 2\rangle$, $\langle 4, 3, 2\rangle$, and $\langle 5, 3, 2\rangle$—are listed in Table I. The standard names given for the last three finite subgroups correspond to their origins in terms of the rotational symmetries of specific 3D regular solids in R^3. With regard to the importance of these three groups in mathematics, a quote from a recent paper by Kostant (1984) reveals their true value:

> The ancient Greeks, especially the school of Plato, had great reverence for the regular polygons in the plane and regular solids in 3-space. The latter—the tetrahedron, cube, octahedron, dodecahedron, and the icosahedron—are often

Table I. Finite Subgroups of the Quaternion Group **Q**

Quaternion group	Symbol	Order
Cyclic	C_n	n
Dicyclic	$\langle p, 2, 2\rangle$	$4p$
Binary tetrahedral	$\langle 3, 3, 2\rangle$	24
Binary octahedral	$\langle 4, 3, 2\rangle$	48
Binary icosahedral	$\langle 5, 3, 2\rangle$	120

referred to as the Platonic solids. The Greeks believed that these regular figures were fundamental in the structure of the universe. If symmetry or its mathematical companion—group theory—is fundamental in the structure of the world, then one of the points of our lecture is the statement that the Greeks were absolutely right. That is, what we will be saying is that in a very profound way, the finite groups of symmetries in 3-space "see" the simple Lie groups (and hence literally Lie theory) in all dimensions.

The rotations of the five familiar solid objects in R^3 listed by Kostant correspond to the operations in the three binary polyhedral groups $\langle 3, 3, 2\rangle$, $\langle 4, 3, 2\rangle$, and $\langle 5, 3, 2\rangle$, i.e., the direct product of the subgroup (p, q, r) of proper rotations with the other half of the group consisting of improper rotations formed by a rotation from (p, q, r) followed by the inversion from C_i. The three binary polyhedral groups will be shown to correspond directly to the three families of leptons.

In order to account for the quarks, one requires the finite rotational groups $[p, q, r]$ in the whole space R^4. The four groups correspond (Coxeter, 1974) to the rotations of the six 4D regular polytopes $\{p, q, r\}$. In Table II these groups are listed along with the 4D regular polytopes associated with the group. Five of the polytopes, the ones associated with groups [3, 3, 4], [3, 4, 3], and [3, 3, 5], have inversion symmetry and therefore their groups are the direct product of the rotation group $[p, q, r]^+$ and the inversion group C_i. The 4D regular simplex $\{3, 3, 3\}$ does not possess inversion symmetry, so one must construct the direct product group with inversion symmetry $[3, 3, 3]^* = [3, 3, 3]^+ \times C_i$, where $[3, 3, 3]^+$ is the

Table II. The 4D Finite Rotation Groups and Their Regular Polytopes

Regular polytope	Symbol	Order
Self-dual regular simplex $\{3, 3, 3\}$	$[3, 3, 3]^*$	120
16-cell $\{3, 3, 4\}$ and 24-cell $\{4, 3, 3\}$	[3, 3, 4]	384
Self-dual 24-cell $\{3, 4, 3\}$	[3, 4, 3]	1152
600-cell $\{3, 3, 5\}$ and 120-cell $\{5, 3, 3\}$	[3, 3, 5]	14400

normal rotation group of the 4D regular simplex. The four polytope groups will be shown to correspond directly to four quark families.

The regular 4D and the 3D geometrical objects are related. These 4D regular polytopes $\{p, q, r\}$ are built up traditionally by using the quaternions in the *binary polyhedral groups*, and *therein lies the intimate connection among the groups representing the 3D and the 4D regular solids.* The same connection will be the primary means for understanding the physical pairings among the lepton and quark families and will supply the mass ratios of all their particles.

Examples of the more important connections include the 24 quaternions in the binary tetrahedral group $\langle 3, 3, 2\rangle$, which are partitioned into two parts: the 8 quaternions of $\langle 2, 2, 2\rangle$, which are the vertices of $\{3, 3, 4\}$, and the remaining 16 quaternions, which make the $\{4, 3, 3\}$, the symmetries of both polytopes described by the group [3, 3, 4]. The whole group of 24 quaternions in $\langle 3, 3, 2\rangle$ are the 24 vertices of $\{3, 4, 3\}$, which corresponds to the group [3, 4, 3]. The 120 quaternions of $\langle 5, 3, 2\rangle$ are the 120 vertices of $\{3, 3, 5\}$, which corresponds to [3, 3, 5], while a select subset of them make the regular simplex $\{3, 3, 3\}$. These relationships will reappear in a later section to help determine the invariant polynomial bases for the 4D groups.

9. THE POLYNOMIAL BASIS AND THE MASS RATIOS

The primary motivation for examining the finite rotational subgroups in R^3 and R^4 of the minimal Standard Model gauge group lies in the identification of the particle families and their mass spectrum. One knows that the mass must be an invariant under all operations of the internal symmetry group. However, the group can be continuous *or finite* as long as it is "$O(3)$-like" in order to ensure (Wigner, 1939) invariance under space-time transformations. All the selected finite rotational groups in R^3 and R^4 will meet this relativistic requirement.

Before proceeding to the pertinent properties of these finite rotational groups, I would like to mention a surprise twist that nature seems to have chosen. Even though I show that the finite *rotational* groups dictate the family structure, mathematicians would be strong proponents for the finite *reflection* groups in the two spaces R^3 and R^4 because reflections are more fundamental, i.e., rotations result from successive reflections. However, an investigation of the mathematical properties of both types of groups indicates that the finite rotational groups are the fundamental groups chosen by nature for the fermions, because one cannot reproduce the correct mass ratios with the reflection groups. If the leptons and quarks in the Standard Model should actually be composite entities, then the potential role of the finite reflection groups at the compositeness scale should be investigated.

One begins with the polynomial basis for each finite rotation group and uses the invariant rational functions of these polynomials to determine the mass ratios. The derivation of the polynomials for the polyhedral groups can be examined in the references given (Klein, 1956; Sansone and Gerretsen, 1969). I will merely adopt the polynomials and the invariant functions in order to proceed directly toward determining the mass ratios. Therefore, I shall be using results that mathematicians have known since the late 1800s, that each of the finite 3D rotational groups possesses three invariant polynomials which are not all independent because they are related by a syzygy which expresses one of them in terms of the other two polynomials. The two independent polynomials for each finite rotation group then form a polynomial basis which can be written in terms of the u and v that span the unitary plane.

Beginning with the binary polyhedral subgroups of **Q**, one learns that each finite binary polyhedral group $\langle p, q, r\rangle$ has exactly two independent homogeneous polynomials w_1 and w_2 which are written (Klein, 1932, 1956; Sansone and Gerretsen, 1969) in terms of the two complex basis functions z_1 and z_2 of C^2. (N.B. One could use u and v here, but since they have been identified as the particle states, the discussion remains more general in terms of z_1 and z_2.) These two polynomials form an integrity basis, i.e., a polynomial basis, for each finite rotation group. In general, each n-dimensional representation for each group will have a different set of polynomials for the integrity basis (Patera *et al.*, 1978; Cummins and Patera, 1988). The two-dimensional representations of the binary polyhedral groups in the unitary plane require two complex polynomials for the integrity basis. The polynomials w_1 and w_2 were originally determined in the 1800s for the binary polyhedral groups in C^2, and each polynomial is an eigenfunction under the operations of the group with eigenvalues $+1$ or -1. They are usually presented in terms of the ratios w_1/w_2 as given in Table III. The

Table III. Invariant Ratios of the Independent Polynomials for the Finite Subgroups of **Q**

Group	Invariant ratio
$\langle 3, 3, 2\rangle$	$\dfrac{w_1}{w_2}=\dfrac{(z_1^4-2i\sqrt{3}z_1^2z_2^2+z_2^4)^3}{(z_1^4+2i\sqrt{3}z_1^2z_2^2+z_2^4)^3}$
$\langle 4, 3, 3\rangle$	$\dfrac{w_1}{w_2}=\dfrac{(z_1^8+14z_1^4z_2^4+z_2^8)^3}{108z_1^4z_2^4(z_1^4-z_2^4)^4}$
$\langle 5, 3, 2\rangle$	$\dfrac{w_1}{w_2}=\dfrac{-\{(z_1^{20}+z_2^{20})+228(z_1^{15}z_2^5-z_1^5z_2^{15})-494z_1^{10}z_2^{10}\}^3}{1728\{z_1z_2(z_1^{10}+11z_1^5z_2^5-z_2^{10})\}^5}$

constant numerical factors 1, 108, and 1728 in the denominators have been inserted to meet the mathematical requirements that are discussed in the next paragraph. These numerical factors will determine the mass ratios for the particles.

An important operation of complex analysis maps the points on a z-sphere (divided into fundamental regions) to the corresponding points on a Riemann w-sphere (having many layers). The z-spheres for the binary polyhedral groups possess triangular fundamental regions. Defining $z = z_1/z_2$ and the *rational* function $w = w_1/w_2$, one finds that the mapping between the spheres is not unique, i.e., at least two different mappings are possible. Felix Klein investigated the pertinent mathematical properties of the mappings in his famous book *Vorlesungen über das Ikosaeder und die Auflösing der Gleichungen vom fünften Grade* [originally published in 1884; for translation see Klein (1956)] in which he determines unique one-to-one mapping conditions. A different numerical constant N for each group must be included in the definition of w in order to ensure the unique mapping. This constant N arises naturally from the syzygy among the invariant polynomials. Recall that the value of N has already been included in the denominators of the ratios presented in Table III. The ratio $w = w_1/w_2$ is invariant under all linear transformations of each group and the mapping is unique with the corresponding values of N: 1, 108, 1728.

The different N value for each group helps identify the lepton family hierarchy and determines the lepton mass ratios. The following argument to justify the connection of the N values to the mass ratios is incomplete, but it seems reasonable and agrees quite well with the empirical values. However, I would prefer to understand the details much better. Further investigation into the fundamental origins of mass "charge" is being continued.

The argument supporting the connection of the N's to the mass ratios relies upon the residues at the poles for the invariant rational functions $w(z)$. Klein points out that the original complex parameter w (without the N included in the denominator) can be set equal to the absolute invariant J of elliptic modular functions (Sansone and Gerretsen, 1969). This absolute invariant J is expressed in terms of a modulus τ, the ratio of two periods ω_1 and ω_2 on a lattice in the complex plane C. Defining $q = e^{2\pi i\tau}$, we have

$$J(\tau) = (1/1728)[q^{-1} + \sum c(n)q^n] \tag{13}$$

with the $c(n)$ all integers (!) and the summation taken from zero to infinity. The important feature is that $J(\tau)$ has a simple pole at $q = 0$. An integral of $J(\tau)$ [or $w(z)$] around a closed path which surrounds the pole therefore equals a constant times the residue at the pole by Cauchy's residue theorem. Such an integral would be required if one desired to determine the effects of a mass within a volume element acting on the outside world, and the calculation would proceed by applying "Gauss' law" for gravitation. The residue for $J(\tau)$ gets multiplied by the appropriate value of N when the rational function $w(z)$ is the integrand for each group. Thus, there exists a direct relationship between the constants N which will appear in the residues of the integral for particles in the different families of leptons and the gravitational effects on the surrounding environment of these particles. Therefore, the mass ratios will be proportional to the N ratios.

Following the prescription, the values of N for the three binary polyhedral groups $\langle 3, 3, 2\rangle$, $\langle 4, 3, 2\rangle$, and $\langle 5, 3, 2\rangle$ produce the mathematical ratios 1:108:1728. Directly comparing these ratio values to the mass ratios of the massive leptons 0.511:105.7:1784 (in MeV/c^2), there is the same general pattern to the values. The neutrinos do not help here because they all have zero masses by the degree-of-freedom arguments discussed earlier. One is tempted to assign each lepton family to its own binary polyhedral group because the "mass ratios" look reasonable and are highly suggestive. Before such a leap can be made, however, one must be able to show that the quark masses can be determined in exactly the same manner when they are assigned to the regular polytope groups $[p, q, 3]$ in R^4.

10. QUARK MASS RATIOS AND FAMILY PAIRINGS

The invariant polynomials for the 4D polytope groups $[p, q, 3]$ are determined by "projecting" (Coxeter, 1974) each 4D polytope from R^4 to the unitary plane to make a *complex* polygon in C^2. Recall that the normal real regular polygon has n vertices on the circle S^1 in the real plane R^2 and n edges of equal length. The complex polygon in the unitary plane C^2 has at least two vertices along each edge and at least two edges at each vertex. The notation tells the facts: the complex polygon $3\{q\}4$ has 3 vertices per edge and 4 edges per vertex, for example. In this notation the familiar real regular polygon would be written as $2\{n\}2$.

Projecting the regular polytope to the unitary plane allows one to look up the polynomials in a table, for according to Coxeter (1974), "all the self-reciprocal complex polygons have regular polytopes for their real counterparts." The regular polytopes for the last three of the four 4D polytope groups have corresponding self-reciprocal complex polygons $p\{q\}p$ in the unitary plane, and the groups for these self-reciprocal complex polygons are subgroups of the polytope groups. Their polynomials listed in Table IV are the same polynomial invariants as the ones given for the three binary polyhedral groups already discussed in connection with the lepton families and therefore these three polytope groups possess the same

Table IV. Invariant Polynomials for the 4D Rotation Groups $[p, q, r]$ in Terms of the Polynomials w_1 and w_2 for the Subgroups of $\mathbf{Q}$ as Determined via Projection to Complex Polygons $p\{q\}p$ in the Unitary Plane[a]

Group $[p, q, r]$	Polytope	Polygon subgroup	Polynomials
[3, 3, 4]	{3, 3, 4}	3[3]3	w_1, w_2 of $\langle 3, 3, 2\rangle$
[3, 4, 3]	{3, 4, 3}	4[3]4	w_1, w_2 of $\langle 4, 3, 2\rangle$
[3, 3, 5]	{3, 3, 5}	5[3]5	w_1, w_2 of $\langle 5, 3, 2\rangle$

[a]From Coxeter (1974).

constants N. This important connection could have been anticipated from the construction of the polytopes via the quaternions in the binary polyhedral groups.

The remaining 4D regular polytope {3, 3, 3} has five vertices, which, when projected to the unitary plane, cannot be a self-reciprocal complex polygon because it has an odd number of vertices. In addition, the {3, 3, 3} does not possess inversion symmetry, so one must force a combination of this polytope with its dual polytope to achieve an object with ten vertices and inversion symmetry in order to possess the [3, 3, 3]* group properties. I have not yet conclusively identified the invariant polynomials for the group [3, 3, 3]*, but preliminary calculations indicate that the invariant polynomials will be associated with a combination of the groups $\langle p, 2, 2\rangle$ and $\langle 3, 3, 2\rangle$. Should a combination of the dicyclic and tetrahedral groups be necessary for [3, 3, 3]*, this amalgam may be the source of the family mixing among the first two quark families. At the present stage, I will take $N = 1/4$, the value for $\langle p, 2, 2\rangle$, as a temporary value.

The pertinent information determined by the mathematical analysis can be gathered together to guide the family pairings. The polynomial bases and the invariant rational functions $w(z)$ for the finite rotational groups lead directly to these important physical consequences:

1. The first quark family (u, d) based on [3, 3, 3]* is unpaired to a lepton family.
2. The quark and lepton family pairings are $(\nu_e, e) \leftrightarrow (c, s)$, $(\nu_\mu, \mu) \leftrightarrow (t, b)$, and $(\nu_\tau, \tau) \leftrightarrow (t', b')$ because the 4D groups [3, 3, 4], [3, 4, 3], and [3, 3, 5] are matched directly to the groups $\langle 3, 3, 2\rangle$, $\langle 4, 3, 2\rangle$, and $\langle 5, 3, 2\rangle$, respectively.
3. The mass ratios of the paired quark families follow the patterns for the mass ratios of the leptons.
4. There are no more families of leptons or quarks because the present assignments have exhausted the supply of finite rotational subgroups in R^3 and R^4.

One must predict three lepton families and four quark families to agree with the numbers of groups in each real space. The mass values shown in Table V are obtained by using the ratios of the N's and the family pairings that the finite rotational groups suggest. In each "mass theory" column, the mass in brackets $[\cdot]$ is taken as the reference mass for the whole column. Because no absolute reference scale for mass value exists, the use of reference masses is the best one can do with the ratios.

How well do the mass predictions fit the empirical values? For the lepton families, the neutrino mass values are taken to be zero by the degrees-of-freedom argument applied in each family. The three massive lepton values are reasonably close to the actual values in MeV/c^2 units even though they range over three orders of magnitude. The large percentage discrepancy for the electron mass is bothersome, and may indicate that there is more to the mass value determination than just the N ratios. Or perhaps the origin of this particular problem lies with the lack of inversion symmetry for the regular tetrahedron. The direct product group $\langle 3, 3, 2\rangle = (3, 3, 2) \times C_i$ actually requires two tetrahedra, the original and its dual, to comply with the group properties. A factor of two may appear here in the calculations for the mass ratio, but this avenue needs further investigation in order to be taken seriously. The important lepton result is the prediction of reasonable mass values.

For the quarks, one notices in Table V that the predicted up- and down-quark masses do not agree with the current mass values between 2 and 9 MeV/c^2 normally used. In particular, the large up-quark mass

Table V. The Family Pairings for Leptons and Quarks and the Mass Values Predicted by the Finite Rotational Group Scheme[a]

	Lepton families				Quark families			
N value	Group	Lepton flavor	Mass theory, MeV/c^2	Mass known, MeV/c^2	Group	Quark flavor	Mass theory, GeV/c^2	Mass known, GeV/c^2
1/4	None				[3, 3, 3]*	u	0.38	0.004
						d	0.011	0.007
1	$\langle 3, 3, 2\rangle$	ν_e	0	$<10^{-5}$	[4, 3, 3]	c	[1.5]	1.5
		e	[1.0]	0.511		s	0.046	0.2
108	$\langle 4, 3, 2\rangle$	ν_μ	0	<1	[3, 4, 3]	t	160	>91
		μ	108	105.7		b	[5]	5
1728	$\langle 5, 3, 2\rangle$	ν_τ	0	<35	[5, 3, 3]	t'	2600	?
		τ	1728	1784		b'	80	?

[a]Mass values in brackets are taken as reference values.

predicted to be about 380 MeV/c^2 is close to being about one-third the proton mass. These discrepancies will need explanation eventually. A good understanding of the origin of quark mixing might be able to resolve these problems, including the predicted strange quark mass of 46 MeV/c^2, which is quite low compared to mass values above 100 MeV/c^2 normally used.

Because the charm and bottom quark masses are taken as references, only three quark mass values remain to be predicted. The predicted top quark mass at about 160 GeV/c^2 is within the remaining allowable range according to empirical search results. The predicted masses that I consider exciting are the fourth quark family b' quark mass value at about 80 GeV/c^2 and the t' quark mass at a whopping 2600 GeV/c^2! The b' should have been produced already at Fermilab, while the t' will be just within the maximum reach of the available energy of the Superconducting Supercollider. If the b' does not exist near 80 GeV/c^2, one can dismiss the finite rotational group modification of the Standard Model as not correct.

11. CHARGE RELATIONSHIPS

Different types of charges characterize the interactions of the fundamental fermions, and each type of charge has been related to rotations in the different spaces. I have determined the mass "charge" values with the help of the invariant polynomials and their rational functions w associated with the rotations of the finite rotational groups in R^3 and R^4. And I have assigned the three values of color charge to the three pairs of orthogonal planes for general 4D rotations. The weak charge was assigned by the Standard Model to the basis states in the unitary plane where the weak interaction rotates the fundamental fermion particle states. One would expect a unified unique connection among these types of charges.

As a first step toward a unified expression, we can improve on the standard "empirical" assignment of weak hypercharge values for the leptons and the quarks by exploiting the close connection of Y to the inversion element I of the group C_i in the normal spinor space and the conjugate spinor space. Instead of assigning Y eigenvalues to make the electric charge values match the empirical results, one can utilize the isomorphism between $SU(2) \times U(1)_Y$ and $SU'(2) = SU(2) \times C_i$ to assign all leptons and quarks the same weak hypercharge eigenvalue, the eigenvalue of the inversion operator I in each spinor space (Altmann, 1986). For $\langle u, v|$ the eigenvalue of I is -1, so the particles should have the eigenvalue $Y = -1$; for antiparticles in $\langle v^*, -u^*|$, their eigenvalue is $Y = +1$.

Second, we can incorporate the color "eigenvalues" into an expression for the electric charge values. Because the Standard Model group $SU(3)_C \times SU(2)_L \times U(1)_Y$ has three components, the expression for the electric charge could be a relationship having three components, one representing each group component. Notice that the mass "charge" will not contribute, because it is not the eigenvalue of a Lie group generator in the gauge group. In the standard basis, $SU(3)$ has (Renton, 1990) the two diagonal generators T_z and Y_C. For $T_z = 0$, there are only three allowed values of Y_C: 0, +2/3, −2/3. I propose to use these three values of Y_C in an expression for the electric charge Q_e of the lepton and quark particle states:

$$Q_e = T_3 + Y/2 + Y_C \tag{14}$$

where T_3 is the $SU(2)$ weak isospin eigenvalue, Y is the weak hypercharge (i.e., inversion operator) eigenvalue, and Y_C is the eigenvalue for the $SU(3)_C$ generator. Using $Y_C = 0$ for leptons and antileptons, +2/3 for quarks, and −2/3 for antiquarks, all fundamental fermions will have the correct electrical charge values via equation (14). A more fundamental geometrical understanding of this simple connection among the charges in terms of the rotations involved will be reported in another article.

12. SYMMETRY BREAKING

Symmetry breaking and gauge fields are essential components of the minimal Standard Model. With the Lie groups in $SU(3)_C \times SU(2)_L \times U(1)_Y$, spontaneous symmetry breaking is done via the Higgs mechanism or by dynamical methods. But the Higgs boson for the minimal model has not been produced in the experiments and can be ruled out for energies less than about 90 GeV. Dynamical symmetry breaking without a Higgs-type particle may be able to rescue the process, but the results are inconclusive.

A third method of symmetry breaking is possible. If the spontaneously broken symmetry is discrete—and this is the case for the finite rotation group modification of the Standard Model—there are no Goldstone bosons to be eliminated by the Higgs mechanism (Coleman, 1985). Each finite rotational subgroup of $SU(2) \times U(1)_Y$ has four generators which behave in the same general way as the four generators of the parent gauge group of the Standard Model. The symmetry breaking reproduces the massive W and Z°, and all the relations of the Glashow–Weinberg–Salam theory still hold true. One simply adopts the mathematical machinery of the Standard Model to produce the same physical characteristics for the electroweak connection.

In order to verify these statements made about the discrete case producing the same symmetry-breaking results as the minimal Standard Model (but without the Higgs boson), one would need to examine carefully the symmetry-breaking process for the finite rotational groups. Ideally, there will be no conflicts with known theoretical and experimental results. A detailed examination is being conducted and will be reported elsewhere.

13. FINAL COMMENTS

The minimal Standard Model has achieved remarkable success in explaining most of the physical behavior of leptons and quarks. The present examination of the geometrical basis has confirmed its robust character. I have examined some of its geometrical properties in three different equivalent internal symmetry spaces—C^2, Q, and R^4— and have uncovered a great variety of interesting relationships which seem to have been adopted by nature for the leptons and quarks. With the help of its finite rotational subgroups, one can explain the origin of (a) the left-handed preference of the weak interaction, (b) color symmetry, (c) the distinction between leptons and quarks, (d) zero neutrino masses, (e) the family hierarchy predicting three lepton families and four quark families and their pairings, (f) the ratios of the particle masses, and (g) the discrete symmetry breaking. The present investigation has brought about a refreshingly different appreciation for the robustness of the minimal Standard Model as well as some insight into the actual geometrical framework in which nature has built the particle world. Much additional theoretical work needs to be done in order to understand better the connection of leptons and quarks to the finite rotation groups, to elucidate the important details essential for understanding the various charges of the particles, and to examine the discrete symmetry breaking. The acid test, however, remains the production of the b' quark that the modified Standard Model predicts at about 80 GeV/c^2, which would be compelling evidence for the existence of the finite rotational groups at the heart of particle physics.

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